Journal of Geometry and Physics 10 (1993) 155–174
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Holomorphic geometry of non-perturbative superstring theories

Sergey A. Merkulov

Theoretical Problems Department, Russian Academy of Sciences,
Vesnina 12, Moscow 121002, Russian Federation
and Department of Mathematics and Computer Science, Odense University,
Campusvej 55, DK-5230 Odense M, Denmark [1]

Received 27 February 1992

It is shown that the natural complex structure over the loop superspace, $\Omega\mathbb{M}^{d|d}_{NS}$, associated to the Neveu–Schwarz superstring, is invariant not only under pure rotations (S^1) but also under the less evident symmetry group $OSp(1|2) \subset$ Superdiff S^1. [Moreover, it is proved that there is a unique Lorentz and $OSp(1|2)$ invariant complex structure on $\Omega\mathbb{M}^{d|d}_{NS}$.] This result implies that the superspace of all admissible complex structures over $\Omega\mathbb{M}^{d|d}$ is isomorphic to the homogeneous Kähler supermanifold Superdiff $S^1/OSp(1|2)$ rather than to Superdiff S^1/S^1 as was stated by Harari et al. [Nucl. Phys. B 294 (1987) 556] and Zhao et al. [Phys. Lett. B 199 (1987) 37]. The Ricci curvature for Superdiff $S^1/OSp(1|2)$ is calculated. Applying the method of geometric quantization to the Neveu–Schwarz superstring, we construct a representation of superstring vacua in terms of antiholomorphic and horizontal sections of a certain vector bundle over Superdiff $S^1/OSp(1|2)$; it is proved that such sections exist only in dimension 10. We also perform a geometric quantization of the Ramond superstring theory. Again our conclusions do not match with those of Harari et al., because we use a different complex structure on the loop superspace $\Omega\mathbb{M}^{d|d}_{R}$ associated with Ramond superstrings; this complex structure is responsible for the fermionic nature of the corresponding vacuum states.

Keywords: *geometric quantization, superstring*
1991 MSC: 81 T 30, 81 S 10

1. Introduction

Applying methods of geometric quantization to the phase space for open bosonic strings, Bowick and Rajeev [1] found a new purely geometric and *non-perturbative* formulation of string field theories. The cornerstone in their construction is the space, $\mathcal{M}$, of all complex structures over the space of based loops $\Omega R^{d-1,1}$ (consisting of all loops in d-dimensional Minkowski space–time $R^{d-1,1}$

[1] Present address.

passing through the origin), which are transformed into each other by the group Diff S^1. In ref. [1] Bowick and Rajeev identified the space $\mathcal{M}$ with the homogeneous Kähler manifold Diff S^1/S^1 because of the obvious invariance of the natural complex structure over $\Omega R^{d-1,1}$ under the group of rigid rotations (S^1). In ref. [2] we proved, however, that this complex structure, J, is actually invariant under a larger symmetry group $SL(2, R) \subset$ Diff S^1. Moreover, we showed that J is a unique (up to Diff S^1 transformations) Lorentz and $SL(2, R)$ invariant complex structure over $\Omega R^{d-1,1}$. This result leads to the important conclusion that the space of complex structures over $\Omega R^{d-1,1}$ should be identified with the homogeneous Kähler manifold Diff $S^1/SL(2, R)$ (rather than with Diff S^1/S^1), which in turn implies that reparametrization invariant non-perturbative ground states of the open bosonic string are represented as non-trivial antiholomorphic and horizontal sections of the tensor product bundle, $\mathcal{B} \otimes \bar{\Gamma}$, over Diff $S^1/SL(2, R)$, where $\mathcal{B}$ and $\bar{\Gamma}$ are the vacuum Fock bundle and the antiholomorphic canonical bundle, respectively. This result does not change the clear geometric interpretation of the critical dimension made by Bowick and Rajeev [1], since non-trivial antiholomorphic and horizontal sections of $\mathcal{B} \otimes \bar{\Gamma}$ have been proved to exist only in dimension 26 [2].

It is tempting, therefore, to make a more careful analysis of the symmetry group of the natural complex structure over the loop superspace associated to the Neveu–Schwarz or Ramond superstring theory. In section 2 of this paper we show that in the case of the Neveu–Schwarz superstring the complex structure over the corresponding loop superspace, $\Omega \mathbb{M}^{d|d}_{NS}$, is invariant not only under the group of rigid rotations (S^1), but also under the less trivial supergroup $OSp(1|2) \subset$ Superdiff S^1. This result leads to the important conclusion that the superspace, $\mathcal{M}$, of all complex structures over $\Omega \mathbb{M}^{d|d}$ should be identified with the homogeneous Kähler supermanifold Superdiff $S^1/OSp(1|2)$ rather than with Superdiff S^1/S^1 as in refs. [3,4]. We calculate the Ricci curvature of Superdiff $S^1/OSp(1|2)$ and, applying methods of geometric quantization to the Neveu–Schwarz superstring along the lines suggested by Bowick and Rajeev, we construct a representation of superstring vacua as non-trivial antiholomorphic and horizontal sections of the tensor product bundle, $\mathcal{B} \otimes \bar{\Gamma}$, over Superdiff $S^1/OSp(1|2)$. It is shown that such sections exist only in dimension 10.

In section 3 we show that both the complex structure on the phase space of open Ramond superstrings and the phase space itself are not defined quite correctly in ref. [3]. Our conclusions are that the fermionic zero mode in the Ramond sector should be retained in the phase space and the complex structure on $\Omega \mathbb{M}^{d|d}_{R}$ should be appropriately modified. The geometric quantization procedure based on the modified complex structure results in a representation of Ramond superstring vacua in terms of antiholomorphic and horizontal sections of the tensor product bundle, $\mathcal{B} \otimes \bar{\Gamma}$, over Superdiff $S^{1|2}/S^{1|2}$. The novelty compared to refs. [3,4] is that these sections furnish a *spinor* representation of the

Lorentz group. Thus the modified complex structure on the modified phase space is responsible for the fermionic nature of Ramond superstring ground states. This conclusion is in full agreement with the results obtained by standard operator methods [5].

2. Geometric quantization of the Neveu–Schwarz superstring

2.1. SYMMETRY GROUP OF THE NATURAL COMPLEX STRUCTURE

Let $\mathbb{M}^{d|0}$ be the usual even $(d|0)$-dimensional Minkowski space and $\mathbb{M}^{0|d} \equiv \Pi\mathbb{M}^{d|0}$ the corresponding odd $(0|d)$-dimensional space, Π being the parity operator [6]. Let us denote by $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ the space of all maps

$$x \oplus \psi : [0, 2\pi] \to \mathbb{M}^{d|d} \equiv \mathbb{M}^{d|0} \oplus \mathbb{M}^{0|d}$$

with periodic boundary conditions in the bosonic coordinates and antiperiodic boundary conditions in the fermionic coordinates. An element of $\mathscr{L}\mathbb{M}^{d|d}$ is a pair (x^μ, ψ^μ) consisting of a bosonic function $x^\mu(\sigma)$ and a fermionic function $\psi^\mu(\sigma)$ which satisfy the boundary conditions $x^\mu(0) = x^\mu(2\pi)$, $\psi^\mu(0) = -\psi^\mu(2\pi)$, $\mu = 0, 1, \ldots, d-1$. We also define the space, $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$, of based superloops as the quotient space $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}/\mathbb{M}^{d|0}$. The space $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ may be identified with the subspace of $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ consisting of all superloops (x, ψ) with bosonic loop x beginning and ending at the origin of $\mathbb{M}^{d|0}$. It has been shown in ref. [3] that the space $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ may be interpreted either as the configuration space for closed Neveu–Schwarz superstrings or as the phase space for open Neveu–Schwarz superstrings.

The infinite dimensional superalgebra, $\underline{\text{Superdiff}}\ \mathrm{S}^1$, associated to the group Superdiff S^1 is given by

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [L_m, G_r] = (\tfrac{1}{2}m - r)G_{m+r},$$
$$[G_r, G_s] = 2L_{r+s}, \tag{2.1}$$

where L_m and G_r are the bosonic and fermionic generators, respectively, of the super-diffeomorphism group (the indices m, n range over the set of integers, while the indices r, s range over the set of half-integers).

Let us consider the vector fields

$$L_m = -\mathrm{i} \int \mathrm{d}\sigma\, \mathrm{e}^{\mathrm{i}m\sigma} \left[\frac{\mathrm{d}x^\mu(\sigma)}{\mathrm{d}\sigma} \frac{\delta}{\delta x^\mu(\sigma)} + \left(\frac{\mathrm{d}\psi^\mu(\sigma)}{\mathrm{d}\sigma} + \mathrm{i}\frac{m}{2}\psi^\mu(\sigma) \right) \frac{\delta}{\delta \psi^\mu(\sigma)} \right],$$

$$G_r = \int \mathrm{d}\sigma\, \mathrm{e}^{\mathrm{i}r\sigma} \left(\psi^\mu(\sigma) \frac{\delta}{\delta x^\mu(\sigma)} + \mathrm{i} \frac{\mathrm{d}x^\mu(\sigma)}{\mathrm{d}\sigma} \frac{\delta}{\delta \psi^\mu(\sigma)} \right), \tag{2.2}$$

on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$. They satisfy the commutation superalgebra (2.1) and provide thus a representation of the superalgebra $\underline{\mathrm{Superdiff}}\,\mathrm{S}^1$ on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$.

If we expand the loops $x^\mu(\sigma)$ and $\psi^\mu(\sigma)$ in their normal modes,

$$x^\mu(\sigma) = \sum_{n\in\mathbb{Z}} x^\mu_n \mathrm{e}^{in\sigma} ,$$

$$\psi^\mu(\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi^\mu_r \mathrm{e}^{ir\sigma} ,$$

then the set of modes $\{x^\mu_n, \psi^\mu_r, n\in\mathbb{Z}, r=\mathbb{Z}+\frac{1}{2}\}$ provides a natural coordinate system on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$, while the set of modes $\{x^\mu_n, \psi^\mu_r, n\in\mathbb{Z}\backslash 0, r\in\mathbb{Z}+\frac{1}{2}\}$ provides a coordinate system on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$. In this chart the generators (2.2) of $\underline{\mathrm{Superdiff}}\,\mathrm{S}^1$ acquire the form

$$L_m = -\sum_{n\in\mathbb{Z}} (n+m) x^\mu_{-n-m} \frac{\partial}{\partial x^\mu_{-n}}$$

$$- \sum_{s\in\mathbb{Z}+\frac{1}{2}} (r+m/2) \psi^\mu_{-s-m} \frac{\partial}{\partial \psi^\mu_{-s}} ,$$

$$G_r = \sum_{n\in\mathbb{Z}} \psi^\mu_{-n-r} \frac{\partial}{\partial x^\mu_{-n}} + \sum_{s\in\mathbb{Z}+\frac{1}{2}} (r+s) x^\mu_{-s-r} \frac{\partial}{\partial \psi^\mu_{-s}} . \tag{2.3}$$

Let J be a Lorentz invariant almost complex structure over $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$. It is best realized as an integral operator,

$$J(W) = \int \mathrm{d}\sigma\,\mathrm{d}\sigma' \bigg([J^{00}(\sigma,\sigma') W^\mu(\sigma') + J^{01}(\sigma,\sigma') \tilde{W}^\mu(\sigma')] \frac{\delta}{\delta x^\mu(\sigma)} + [J^{10}(\sigma,\sigma') W^\mu(\sigma') + J^{11}(\sigma,\sigma') \tilde{W}^\mu(\sigma')] \frac{\delta}{\delta \psi^\mu(\sigma)} \bigg) , \tag{2.4}$$

where

$$W = \int \mathrm{d}\sigma \left(W^\nu(\sigma) \frac{\delta}{\delta x^\nu(\sigma)} + \tilde{W}^\mu(\sigma) \frac{\delta}{\delta \psi^\mu(\sigma)} \right) \tag{2.5}$$

is an arbitrary vector field on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ [the integral in (2.4) is understood in the sense of the principal value, i.e., $\lim_{\epsilon\to 0}(\int_0^{\sigma-\epsilon} + \int_{\sigma+\epsilon}^{2\pi})$]. Then the condition

$$J^2 = -\mathrm{id} \tag{2.6}$$

takes the form

$$\int \mathrm{d}\sigma' \, [J^{00}(\sigma,\sigma')\, J^{00}(\sigma',\sigma'') + J^{01}(\sigma,\sigma')\, J^{10}(\sigma',\sigma'')] = -\delta(\sigma-\sigma'') ,$$

$$\int \mathrm{d}\sigma' \, [J^{11}(\sigma,\sigma')\, J^{11}(\sigma',\sigma'') + J^{10}(\sigma,\sigma')\, J^{01}(\sigma',\sigma'')] = -\delta(\sigma-\sigma'') ,$$

$$\int \mathrm{d}\sigma' \, J^{01}(\sigma, \sigma') \, [J^{00}(\sigma', \sigma'') + J^{11}(\sigma', \sigma'')] = 0 \, ,$$

$$\int \mathrm{d}\sigma' \, J^{10}(\sigma, \sigma') \, [J^{00}(\sigma', \sigma'') + J^{11}(\sigma', \sigma'')] = 0 \, . \qquad (2.7)$$

Our next task is to find those complex structures on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ which are invariant under the action of a vector field L_k for some $k \in \mathbb{Z}$, or a vector field G_r for some $r \in \mathbb{Z} + \frac{1}{2}$. Such a complex structure has to satisfy the equation

$$\mathscr{L}_{L_k} J = 0 \, , \qquad (2.8)$$

or the equation

$$\mathscr{L}_{G_r} J = 0 \, , \qquad (2.9)$$

respectively, where $\mathscr{L}_W$ stands for the Lie derivative along a vector field W.

To decode the hieroglyphs (2.8) and (2.9) in terms of the kernel (2.4) we note that eqs. (2.8) and (2.9) are equivalent, respectively, to the following equations:

$$J([L_k, V]) = [L_k, J(V)] \, , \qquad (2.10)$$

$$J([G_r, W]) = [G_r, J(W)] \, , \qquad (2.11)$$

which must hold for arbitrary vector fields V and W on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$.

First we compute the commutators

$$\begin{aligned}
[L_k, V] = & -\mathrm{i} \int\int \mathrm{d}\sigma \, \mathrm{d}\sigma' \, \mathrm{e}^{\mathrm{i}k\sigma'} \left\{ \frac{\mathrm{d}x^\nu(\sigma')}{\mathrm{d}\sigma'} \frac{\delta V^\mu(\sigma)}{\delta x^\nu(\sigma')} \right. \\
& \left. + \left[\left(\frac{\mathrm{d}}{\mathrm{d}\sigma'} + \mathrm{i}\frac{k}{2} \right) \psi^\mu(\sigma') \right] \frac{\delta V^\nu(\sigma)}{\delta x^\mu(\sigma')} \right\} \frac{\delta}{\delta x^\nu(\sigma)} \\
& -\mathrm{i} \int\int \mathrm{d}\sigma \, \mathrm{d}\sigma' \, \mathrm{e}^{\mathrm{i}k\sigma'} \left\{ \frac{\mathrm{d}x^\nu(\sigma')}{\mathrm{d}\sigma'} \frac{\delta \tilde{V}^\mu(\sigma)}{\delta x^\nu(\sigma')} \right. \\
& \left. + \left[\left(\frac{\mathrm{d}}{\mathrm{d}\sigma'} + \mathrm{i}\frac{k}{2} \right) \psi^\mu(\sigma') \right] \frac{\delta \tilde{V}^\nu(\sigma)}{\delta x^\mu(\sigma')} \right\} \frac{\delta}{\delta \psi^\nu(\sigma)} \\
& +\mathrm{i} \int\int \mathrm{d}\sigma \quad \mathrm{e}^{\mathrm{i}k\sigma} \left\{ \frac{\mathrm{d}V^\nu(\sigma)}{\mathrm{d}\sigma} \frac{\delta}{\delta x^\nu(\sigma)} \right. \\
& \left. + \left[\left(\frac{\mathrm{d}}{\mathrm{d}\sigma} + \mathrm{i}\frac{k}{2} \right) \tilde{V}^\nu(\sigma) \right] \frac{\delta}{\delta \psi^\nu(\sigma)} \right\} ,
\end{aligned}$$

and

$$[G_r, W] = \int\int \mathrm{d}\sigma\,\mathrm{d}\sigma'\,\mathrm{e}^{\mathrm{i}r\sigma'}\left(\mathrm{i}\frac{\mathrm{d}x^\nu(\sigma')}{\mathrm{d}\sigma'}\frac{\delta W^\mu(\sigma)}{\delta\psi^\nu(\sigma')} + \psi^\mu(\sigma')\frac{\delta W^\nu(\sigma)}{\delta x^\mu(\sigma')}\right)\frac{\delta}{\delta x^\mu(\sigma)}$$
$$+ \int\int \mathrm{d}\sigma\,\mathrm{d}\sigma'\,\mathrm{e}^{\mathrm{i}r\sigma'}\left(\mathrm{i}\frac{\mathrm{d}x^\nu(\sigma')}{\mathrm{d}\sigma'}\frac{\delta \tilde{W}^\mu(\sigma)}{\delta\psi^\nu(\sigma')} + \psi^\mu(\sigma')\frac{\delta \tilde{W}^\nu(\sigma)}{\delta x^\mu(\sigma')}\right)\frac{\delta}{\delta \psi^\nu(\sigma)}$$
$$- \int\int \mathrm{d}\sigma\,\mathrm{e}^{\mathrm{i}r\sigma}\left(\mathrm{i}\frac{\mathrm{d}W^\nu(\sigma)}{\mathrm{d}\sigma}\frac{\delta}{\delta\psi^\nu(\sigma)} + \tilde{W}^\nu(\sigma)\frac{\delta}{\delta x^\nu(\sigma)}\right),$$

where we used the identities

$$\frac{\delta^2}{\delta x^\mu(\sigma)\,\delta x^\nu(\sigma')} = \frac{\delta^2}{\delta x^\nu(\sigma')\,\delta x^\mu(\sigma)}, \qquad \frac{\delta^2}{\delta x^\mu(\sigma)\,\delta \psi^\nu(\sigma')} = \frac{\delta^2}{\delta \psi^\nu(\sigma')\,\delta x^\mu(\sigma)},$$
$$\frac{\delta^2}{\delta \psi^\mu(\sigma)\,\delta \psi^\nu(\sigma')} = -\frac{\delta^2}{\delta \psi^\nu(\sigma')\,\delta \psi^\mu(\sigma)}.$$

It is now straightforward but tedious to check that eq. (2.10) holds for any V provided the almost complex structure (2.4) satisfies the equations

$$\mathrm{e}^{\mathrm{i}k\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{00}(\sigma,\sigma') + \frac{\mathrm{d}}{\mathrm{d}\sigma'}[J^{00}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}k\sigma'}] = 0,$$
$$\mathrm{e}^{\mathrm{i}k\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{01}(\sigma,\sigma') + \left(\frac{\mathrm{d}}{\mathrm{d}\sigma'} - \mathrm{i}\frac{k}{2}\right)[J^{00}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}k\sigma'}] = 0,$$
$$\mathrm{e}^{\mathrm{i}k\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma} + \mathrm{i}\frac{k}{2}\right)J^{10}(\sigma,\sigma') + \frac{\mathrm{d}}{\mathrm{d}\sigma'}[J^{10}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}k\sigma'}] = 0,$$
$$\mathrm{e}^{\mathrm{i}k\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma} + \mathrm{i}\frac{k}{2}\right)J^{11}(\sigma,\sigma') + \left(\frac{\mathrm{d}}{\mathrm{d}\sigma'} - \mathrm{i}\frac{k}{2}\right)[J^{11}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}k\sigma'}] = 0, \qquad (2.12)$$

while eq. (2.11) is satisfied for any W if and only if

$$\mathrm{e}^{\mathrm{i}r\sigma}\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{00}(\sigma,\sigma') = \frac{\mathrm{d}}{\mathrm{d}\sigma'}[J^{11}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}r\sigma'}],$$
$$\mathrm{e}^{\mathrm{i}r\sigma}J^{11}(\sigma,\sigma') = J^{00}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}kr'},$$
$$\mathrm{e}^{\mathrm{i}r\sigma}J^{10}(\sigma,\sigma') = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\sigma'}[J^{01}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}kr'}],$$
$$\mathrm{e}^{\mathrm{i}r\sigma}J^{01}(\sigma,\sigma') = J^{10}(\sigma,\sigma')\,\mathrm{e}^{\mathrm{i}kr'}. \qquad (2.13)$$

Thus eqs. (2.12) [eqs. (2.13)] provide the necessary and sufficient conditions for the almost complex structure (2.4) to be invariant under the action of the vector field L_k, $k\in\mathbb{Z}$ [G_r, $r\in\mathbb{Z}+\frac{1}{2}$].

If $k=0$, then eqs. (2.12) take the form

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{00}(\sigma,\sigma')+\frac{\mathrm{d}}{\mathrm{d}\sigma'}J^{00}(\sigma,\sigma')=0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{01}(\sigma,\sigma')+\frac{\mathrm{d}}{\mathrm{d}\sigma'}J^{01}(\sigma,\sigma')=0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{10}(\sigma,\sigma')+\frac{\mathrm{d}}{\mathrm{d}\sigma'}J^{10}(\sigma,\sigma')=0\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{11}(\sigma,\sigma')+\frac{\mathrm{d}}{\mathrm{d}\sigma'}J^{11}(\sigma,\sigma')=0\,,$$

which imply

$$J^{00}(\sigma,\sigma')=J^{00}(\sigma-\sigma')\,,\qquad J^{01}(\sigma,\sigma')=J^{01}(\sigma-\sigma')\,,$$
$$J^{10}(\sigma,\sigma')=J^{10}(\sigma-\sigma')\,,\qquad J^{11}(\sigma,\sigma')=J^{11}(\sigma-\sigma')\,. \tag{2.14}$$

This is an expected result, since a solution of the equation $\mathscr{L}_{L_0}J=0$ must be invariant under rigid rotations (S^1).

Let us now find conditions on an almost complex structure J over $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ which ensure the invariance of J under the actions of both the vector field L_0 and a vector field G_r for some $r\in\mathbb{Z}+\frac{1}{2}$. From (2.7), (2.13) and (2.14) it follows that such an almost complex structure, which we denote by J_r, necessarily satisfies the equations

$$J^{01}_r(\sigma-\sigma')=0\,,\qquad J^{10}_r(\sigma-\sigma')=0\,,$$
$$J^{11}_r(\sigma,\sigma')=\mathrm{e}^{-\mathrm{i}r(\sigma-\sigma')}J^{00}_r(\sigma-\sigma')\,,$$
$$(\mathrm{e}^{2\mathrm{i}r(\sigma-\sigma')}-1)\frac{\mathrm{d}}{\mathrm{d}\sigma}J^{00}_r(\sigma-\sigma')=2\mathrm{i}rJ^{00}_r(\sigma-\sigma')\,,\quad r\in\mathbb{Z}+\tfrac{1}{2}\,. \tag{2.15}$$

The latter equation is solved by

$$J^{00}_r(\sigma-\sigma')=A(1-\mathrm{e}^{2\mathrm{i}r(\sigma-\sigma')})^{-1}$$
$$=-\tfrac{1}{2}A\,\mathrm{e}^{2\mathrm{i}r(\sigma-\sigma')}[1+\mathrm{i}\cot\mathrm{an}(r\sigma)]\,,$$

for some constant A. It is easy to check that the functions $J^{00}_r(\sigma-\sigma')$ and $J^{11}_r(\sigma-\sigma')$ with $r=\pm\frac{1}{2}$ can be normalized so as to satisfy eqs. (2.7). Therefore we conclude that there exists a unique almost complex structure $J_{\pm 1/2}$ over $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ which has the kernel

$$J^{00}_{+1/2}(\sigma-\sigma')=-\mathrm{i}\mathrm{e}^{\mathrm{i}(\sigma-\sigma')}[1+\mathrm{i}\cot\mathrm{an}\,\tfrac{1}{2}(\sigma-\sigma')]$$
$$=-\mathrm{i}\sum_{m\geqslant 1}\mathrm{e}^{\mathrm{i}m\sigma}+\mathrm{i}\sum_{m<1}\mathrm{e}^{\mathrm{i}m\sigma}\,,$$

$$J^{11}_{+1/2}(\sigma-\sigma')=-\mathrm{i}e^{(\mathrm{i}/2)(\sigma-\sigma')}[1+\mathrm{i}\cot\!\mathrm{an}\,\tfrac{1}{2}(\sigma-\sigma')]$$
$$=-\mathrm{i}\sum_{r>0}e^{\mathrm{i}r\sigma}+\mathrm{i}\sum_{r<0}e^{\mathrm{i}r\sigma},$$
$$J^{01}_{+1/2}(\sigma-\sigma')=0, \qquad J^{10}_{+1/2}(\sigma-\sigma')=0,$$

and is invariant under the actions of L_0 and $G_{+1/2}$, and there is also a unique complex structure $J_{-1/2}$ on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ which has the kernel

$$J^{00}_{-1/2}(\sigma-\sigma')=-\mathrm{i}[1+\mathrm{i}\cot\!\mathrm{an}\,\tfrac{1}{2}(\sigma-\sigma')]$$
$$=-\mathrm{i}\sum_{m\geq 0}e^{\mathrm{i}m\sigma}+\mathrm{i}\sum_{m<0}e^{\mathrm{i}m\sigma},$$
$$J^{11}_{-1/2}(\sigma-\sigma')=-\mathrm{i}e^{(\mathrm{i}/2)(\sigma-\sigma')}[1+\mathrm{i}\cot\!\mathrm{an}\,\tfrac{1}{2}(\sigma-\sigma')]$$
$$=-\mathrm{i}\sum_{r>0}e^{\mathrm{i}r\sigma}+\mathrm{i}\sum_{r<0}e^{\mathrm{i}r\sigma},$$
$$J^{01}_{-1/2}(\sigma-\sigma')=0, \qquad J^{10}_{-1/2}(\sigma-\sigma')=0,$$

and is invariant under actions of L_0 and $G_{-1/2}$. From the commutation superalgebra (2.1) it follows that $J_{+1/2}$ must also satisfy the equation $\mathscr{L}_{L_1}J_{+1/2}=0$, while $\mathscr{L}_{L_{-1}}J_{-1/2}=0$. These facts can be checked directly with the help of eqs. (2.12).

Thus we constructed a pair of almost complex structures, $J_{+1/2}$ and $J_{-1/2}$, on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ which are unique solutions of the following equations:

$$\mathscr{L}_{L_0}J_{+1/2}=\mathscr{L}_{G_{1/2}}J_{+1/2}=\mathscr{L}_{L_1}J_{+1/2}=0,$$
$$\mathscr{L}_{L_0}J_{-1/2}=\mathscr{L}_{G_{-1/2}}J_{-1/2}=\mathscr{L}_{L_{-1}}J_{-1/2}=0.$$

The complex structure $J_{+1/2}$ when applied to a tangent vector

$$W=\int d\sigma\left(W^{\nu}(\sigma)\frac{\delta}{\delta x^{\nu}(\sigma)}+\tilde{W}^{\mu}(\sigma)\frac{\delta}{\delta\psi^{\mu}(\sigma)}\right)$$

with

$$W^{\mu}(\sigma)=\sum_{n\in\mathbb{Z}}W^{\mu}_{n}e^{\mathrm{i}n\sigma}, \qquad \tilde{W}^{\mu}=\sum_{r\in\mathbb{Z}+\frac{1}{2}}W^{\mu}_{r}e^{\mathrm{i}r\sigma},$$

gives a tangent vector $J_{+1/2}(W)$ with components

$$(J_{+1/2}(W))^{\mu}=+\mathrm{i}W^{\mu}_{0}-\mathrm{i}\sum_{n\neq 0}\mathrm{sgn}(n)\,W^{\mu}_{n}e^{\mathrm{i}n\sigma},$$
$$J_{+1/2}(W))^{\mu}=-\mathrm{i}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\mathrm{sgn}(r)\,\tilde{W}^{\mu}_{r}e^{\mathrm{i}r\sigma},$$

while

$$(J_{-1/2}(W))^{\mu}=-\mathrm{i}W^{\mu}_{0}-\mathrm{i}\sum_{n\neq 0}\mathrm{sgn}(n)\,W^{\mu}_{n}e^{\mathrm{i}n\sigma},$$

$$(J_{-1/2}(W))^{\mu} = -\mathrm{i} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \mathrm{sgn}(r)\ \tilde{W}^{\mu}_{r}\ \mathrm{e}^{\mathrm{i}r\sigma}\,.$$

The crucial point of our discussion is that the $(L_0, L_1, G_{1/2})$-invariant almost complex structure $J_{+1/2}$ and the $(L_0, L_{-1}, G_{-1/2})$-invariant almost complex structure $J_{-1/2}$ differ from each other only by their actions on the zero mode W^{μ}_{0}. This difference becomes irrelevant when we go to the quotient superspace $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}/\mathbb{M}^{d|0}$. Thus we conclude that the almost complex structures $J_{+1/2}$ and $J_{-1/2}$ induce one and the same integrable almost complex structure,

$$J(W) = -\mathrm{i} \sum_{n\neq 0} \mathrm{sgn}(n)\ W^{\mu}_{n}\ \partial/\partial x^{\mu}_{n} - \mathrm{i} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \mathrm{sgn}(r)\ \tilde{W}^{\mu}_{r}\ \partial/\partial\psi^{\mu}_{r}\,, \tag{2.16}$$

on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$, which coincides exactly with the natural complex structure used in ref. [3].

Another conclusion is that the complex structure J is invariant under the action of the subgroup $\mathrm{OSp}(1|2)$ of Superdiff S^1 generated by the vector fields L_{-1}, L_0, L_1, $G_{1/2}$ and $G_{-1/2}$, and this is a unique complex structure on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ having this property. This result implies that the supermanifold, $\mathscr{M}$, of all complex structures on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ which are connected to each other by Superdiff S^1 is isomorphic to the homogeneous supermanifold Superdiff $\mathrm{S}^1/\mathrm{OSp}(1|2)$ rather than to Superdiff $\mathrm{S}^1/\mathrm{S}^1$ as was claimed in refs. [3,4].

2.2. RICCI CURVATURE OF Superdiff $\mathrm{S}^1/\mathrm{OSp}(1|2)$

Let us now compute the Ricci curvature of $\mathscr{M}$ = Superdiff $\mathrm{S}^1/\mathrm{OSp}(1|2)$. Let K_m, $m\in\mathbb{Z}$, and H_r, $r\in\mathbb{Z}+\frac{1}{2}$ be generators of the Lie superalgebra of the superdiffeomorphism group of the circle, Superdiff S^1. We use the same symbols to denote the corresponding left invariant vector fields on Superdiff S^1, which, by construction, satisfy the commutation superalgebra

$$[K_m, K_n] = (m-n)K_{m+n}\,, \qquad [K_m, H_r] = (\tfrac{1}{2}m - r)H_{m+r}\,,$$
$$[H_r, H_s] = 2K_{r+s}\,, \tag{2.17}$$

at the origin. A general tangent vector to $\mathscr{M}$ at its origin is a linear combination

$$\theta = \sum_{n\neq\pm 1,0} \theta_n K_n + \sum_{r\neq +\frac{1}{2}} \theta_r H_r\,.$$

The natural almost complex structure $\tilde{J}$ at the origin of $\mathscr{M}$ is defined by [3,4]

$$\tilde{J}(\theta) = \sum_{n\in\neq\pm 1,0} -\mathrm{i}\ \mathrm{sgn}(n)\ \theta_n L_n + \sum_{r\neq\pm\frac{1}{2}} -\mathrm{i}\ \mathrm{sgn}(r)\ \theta_r G_r\,. \tag{2.18}$$

At other points of $\mathscr{M}$ the almost complex structure is defined by left translations. From the theory of invariant almost complex structures (see, e.g., ch. 10, §6 in ref. [7]) and the commutation relations (2.17) it follows that the almost complex structure $\tilde{J}$ is integrable. This establishes that $\mathcal{M}$ is a complex supermanifold.

There exists a *unique* homogeneous Kähler form ω on $\mathcal{M}$. By homogeneity, one needs to determine ω only at the origin of $\mathcal{M}$. Upon computation we find that the closeness condition $d\omega = 0$ implies the following explicit expressions (cf. refs. [3,4]):

$$\omega(K_m, K_n) = a(m^3 - m)\delta_{m,-n} ,$$

$$\omega(K_m, H_r) = \omega(H_r, K_m) = 0 ,$$

$$\omega(H_r, H_s) = a(4r^2 - 1)\delta_{r,-s} , \tag{2.19}$$

where a is a non-zero parameter and m, $n \in \mathbb{Z} \setminus \{\pm 1, 0\}$, r, $s \in \{\mathbb{Z} + \frac{1}{2}\} \setminus \{\pm \frac{1}{2}\}$.

The data (J, ω) constitute a Kähler structure on $\mathcal{M}$, and our next task is to compute the corresponding Ricci tensor using the method proposed by Freed [8].

First we note that the complex structure J gives the following decomposition of the complexified Lie superalgebra $\mathbb{C} \otimes \underline{\text{Superdiff } S^1}$ (at the origin):

$$\mathbb{C} \otimes \underline{\text{Superdiff } S^1} = \hat{\mathcal{M}}_+ + \hat{\mathcal{M}}_- + \mathbb{C} \otimes \underline{\text{OSp}(1|2)} ,$$

where $\hat{\mathcal{M}}_+$ is spanned by the generators K_{-m}, H_{-r} ($m \geqslant 2$, $r \geqslant \frac{3}{2}$) and $\hat{\mathcal{M}}_-$ is spanned by K_m, H_r ($m \geqslant 2$, $r \geqslant \frac{3}{2}$).

Next we define the Toeplitz $\mathcal{O}_{\mathcal{M}}$-linear operator φ by writing

$$\varphi(X) = \nabla_X - \mathcal{L}_X , \tag{2.20}$$

where ∇_X stands for the covariant Kähler derivative and $\mathcal{L}_X$ the Lie derivative along a vector field X. Then the Riemann tensor is given by

$$R(X, Y) = [\varphi(X), \varphi(Y)] - \varphi([X, Y]) . \tag{2.21}$$

Thus, in order to compute the Riemann tensor one should know explicit expressions for the Toeplitz operators. According to the theorem of Freed [8], they are given by

$$\varphi(X_-) = -\pi_+ \circ \text{ad } X_- \quad \text{for } X_- \in \hat{\mathcal{M}}_- ,$$

$$\varphi(X_+) = -\varphi(\bar{X}_+)^+ \qquad \text{for } X_+ \in \hat{\mathcal{M}}_+ , \tag{2.22}$$

where ad means an adjoint operator, π_+ is the projection operator onto $\hat{\mathcal{M}}_+$ and $\varphi(\bar{X}_+)^+$ is the adjoint of $\varphi(\bar{X}_+)$ with respect to the Kähler metric. From (2.17), (2.19) and (2.22) it is not difficult to find (cf. refs. [3,4])

$$\varphi(K_m)K_{-p} = -\theta(p - m - 1)(p + m)K_{m-p} ,$$

$$\varphi(K_m)H_{-r} = -\theta(r - m - \tfrac{1}{2})(r + \tfrac{1}{2}m)H_{m-r} ,$$

$$\varphi(K_{-m})K_{-p} = (p + 2m)\frac{\omega(p)}{\omega(p+m)}K_{-m-p} ,$$

$$\varphi(K_{-m})H_{-r}=(r+\tfrac{3}{2}m)\,\frac{\epsilon(r)}{\epsilon(r+m)}\,H_{-r-m}\,,$$

$$\varphi(H_r)K_{-p}=-\theta(p-r-\tfrac{1}{2})(r+\tfrac{1}{2}p)H_{r-p}\,,$$

$$\varphi(H_r)H_{-s}=-2\theta(s-r-1)K_{r-s}\,,$$

$$\varphi(H_{-r})K_{-p}=2\,\frac{\omega(p)}{\epsilon(p+r)}\,H_{-p-r}\,,$$

$$\varphi(H_{-r})H_{-s}=\tfrac{1}{2}(s+3r)\,\frac{\epsilon(s)}{\epsilon(r+s)}\,K_{-r-s}\,,$$

where $m\geqslant 2$, $r\geqslant\frac{3}{2}$, $\omega(p)\equiv p^3-p$ and $\epsilon(r)\equiv 4r^2-1$.

Using (2.22) it is now straightforward to compute the Riemann tensor, whose components are defined by

$$R(K_{-m},K_n)K_{-p}=R_{\bar{m}np}{}^{q}\,K_{-q}\,,\qquad R(K_{-m},K_n)H_{-r}=R_{\bar{m}nr}{}^{s}\,H_{-s}\,,$$

$$R(H_{-r},H_s)K_{-p}=R_{\bar{r}sp}{}^{q}\,K_{-q}\,,\qquad R(H_{-r},H_s)H_{-t}=R_{\bar{r}st}{}^{u}\,H_{-u}\,.$$

The diagonal terms of the Riemann tensor are given by (there is no summation over repeated indices)

$$R_{\bar{m}np}{}^{p}=\left(-\theta(p-m-1)(p+m)^2\,\frac{\omega(p-m)}{\omega(p)}\right.$$
$$\left.+(p+2m)^2\,\frac{\omega(p)}{\omega(p+m)}-2mp\right)\delta_{m,n}\,,$$

$$R_{\bar{m}nt}{}^{t}=\left(-\theta(t-m-\tfrac{1}{2})(t+\tfrac{1}{2}m)^2\,\frac{\epsilon(t-m)}{\epsilon(t)}\right.$$
$$\left.+(t+\tfrac{3}{2}m)^2\,\frac{\epsilon(t)}{\epsilon(t+m)}-2mt\right)\delta_{m,n}\,,$$

$$R_{\bar{r}sp}{}^{p}=-\left(\theta(p-r-\tfrac{1}{2})(r+\tfrac{1}{2}p)^2\,\frac{\epsilon(r-p)}{\omega(p)}+\frac{4\omega(p)}{\epsilon(p+r)}-2p\right)\delta_{r,s}\,,$$

$$R_{\bar{r}st}{}^{t}=-\left(4\theta(t-r-1)\,\frac{\epsilon(t-r)}{\epsilon(t)}+\tfrac{1}{4}(t+3r)^2\,\frac{\epsilon(t)}{\omega(t+r)}-2t\right)\delta_{r,s}\,.$$

The Ricci tensor would be given by the supertrace of the Riemann tensor over $p\geqslant 2$ and $t\geqslant\frac{3}{2}$. Just as in the case of Superdiff S^1/S^1 [3,4], we find that the corresponding sum is convergent without need of a regularization scheme. The results are

$$\mathrm{Ric}(K_{-m},K_n)=-\tfrac{10}{8}(m^3-m)\delta_{m,n}\,,$$

$$\mathrm{Ric}(G_{-r}, G_s) = -\tfrac{10}{8}(4r^2 - 1)\delta_{r,s}\,. \tag{2.23}$$

2.3. VACUUM FOCK BUNDLE OVER Superdiff $S^1/\mathrm{OSp}(1|2)$

Let us define the symplectic metric [3]

$$\omega(U,V) = \frac{1}{2\pi}\int \eta_{\mu\nu}\left(U^\mu(\sigma)\frac{\mathrm{d}}{\mathrm{d}\sigma}V^\nu(\sigma) - \mathrm{i}\tilde{U}^\mu(\sigma)\tilde{V}^\nu(\sigma)\right)\mathrm{d}\sigma$$
$$= \mathrm{i}\sum_{m\in\mathbb{Z}} mU^\mu_{-m}V^\nu_m\eta_{\mu\nu} - \mathrm{i}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{U}^\mu_r\tilde{V}^\nu_{-r}\eta_{\mu\nu} \tag{2.24}$$

on the phase superspace $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ of the open Neveu–Schwarz superstring. Though this metric is obviously degenerate on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$, it can be regarded as a lift to $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ of a non-degenerate symplectic metric on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$, which we denote by the same symbol ω. Therefore the quotient superspace $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}} = \mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}/\mathbb{M}^{d|0}$ comes equipped with a natural symplectic structure. The vector fields (2.3) on $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}}$ are mapped under the projection $\mathscr{L}\mathbb{M}^{d|d}_{\mathrm{NS}} \to \Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ into vector fields $L_m \bmod(\partial/\partial x^\mu_0)$, $G_r \bmod(\partial/\partial x^\mu_0)$ on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$, which satisfy the same commutation superalgebra (2.1) and are denoted by the same symbols. These vector fields are Hamiltonian vector fields (the notations used in this subsection are explained in the appendix),

$$L_n = X_{\lambda_n}\,, \qquad G_r = X_{\varphi_r}\,,$$

corresponding to the following functionals:

$$\lambda_n = \frac{1}{4\pi}\int \mathrm{d}\sigma\, \mathrm{e}^{\mathrm{i}n\sigma}\left[\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}x(\sigma)\right)^2 + \mathrm{i}\varphi(\sigma)\frac{\mathrm{d}}{\mathrm{d}\sigma}\varphi(\sigma)\right],$$
$$\varphi_r = -\frac{\mathrm{i}}{2\pi}\int \mathrm{d}\sigma\, \mathrm{e}^{\mathrm{i}r\sigma}\varphi(\sigma)\frac{\mathrm{d}}{\mathrm{d}\sigma}x(\sigma)\,, \tag{2.25}$$

on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ (cf. ref. [3]). As follows from (2.1) these functionals satisfy the following Poisson superalgebra:

$$\{\lambda_m, \lambda_n\} = \mathrm{i}(m-n)\lambda_{m+n}\,, \qquad \{\lambda_m, \varphi_r\} = \mathrm{i}(\tfrac{1}{2}m - r)\varphi_{m+r}\,,$$
$$\{\varphi_r, \varphi_s\} = 2\mathrm{i}\lambda_{r+s}\,. \tag{2.26}$$

The Lorentz and $\mathrm{OSp}(1|2)$ invariant complex structure on $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ determines the hermitian inner product on tangent vectors,

$$\omega(U, JU) = \sum_{m\geqslant 1} m|U_m|^2 + \sum_{r\geqslant\frac{1}{2}} \tilde{U}_r\tilde{U}_{-r}\,, \tag{2.27}$$

which implies that $\Omega\mathbb{M}^{d|d}_{\mathrm{NS}}$ comes equipped with a Kähler structure with the Kähler potential given by

$$K = \sum_{n \geq 1} n|x_n|^2 + \sum_{r \geq \frac{1}{2}} \psi_r \psi_{-r} .$$

The complex structure J on $\Omega \mathbb{M}_{NS}^{d|d}$ determines a Kähler polarization (see appendix). From (2.16) it follows that the subspace $T^{1,0}\Omega \mathbb{M}_{NS}^{d|d}$ of the complexified tangent bundle over $\Omega \mathbb{M}_{NS}^{d|d^{sd|d}}$ is spanned by all the vectors $\{\partial/\partial x_n^\mu, \partial/\partial \psi_r^\mu$ with $n>0$ and $r \geq \frac{1}{2}\}$. Thus the quantum Hilbert space H consists of functionals $\Psi: \Omega \mathbb{M}_{NS}^{d|d} \to \mathbb{C}$ which depend only on the coordinates $x_{-n}^\mu, \psi_{-r}^\mu$ with $n \geq 1$ and $r \geq \frac{1}{2}$.

Our next task is to construct quantum operators $r(\lambda_n)$ acting on H, which correspond to the classical "observables" (2.25). The corresponding detailed calculations can be found in ref. [3] and we only sketch the results. Since the operators $\mathcal{O}_{\lambda_n}$ and $\mathcal{O}_{\varphi_r}$ preserve the polarization for $n \geq -1$ and $r \geq -\frac{1}{2}$ and hence map the quantum Hilbert space H onto itself, they are appropriate at the quantum level and one may define

$$r(\lambda_n) = \mathcal{O}_{\lambda_n}, \qquad r(\varphi_r) = \mathcal{O}_{\varphi_r}, \quad n \geq -1, r \geq -1/2 .$$

(In ref. [3] the operator $r(\lambda_0)$ was defined as $\mathcal{O}_\lambda + a_0$, where the constant shift a_0 was introduced with the aim to account for possible normal ordering ambiguities. However, the geometric quantization method used in calculating $r(\lambda_n)$ and $r(\varphi_r)$ has the advantage of avoiding the operator ordering problems, and we set a_0 to zero.)

Following refs. [2,3] we note that physical superstring amplitudes should be invariant under Superdiff S^1. Since the complex structure J used in the construction of the quantum Hilbert space H changes under the action of a general element of Superdiff S^1, we should consider the space $\mathcal{M}$ of all complex structures on $\Omega \mathbb{M}_{NS}^{d|d}$ which are transformed into each other by Superdiff S^1. In subsection 2.1 it is proved that

$$\mathcal{M} = \text{Superdiff } S^1 / \text{OSp}(1|2) .$$

Let us define a Fock vector bundle $\pi: \mathcal{B} \to \mathcal{M}$ over $\mathcal{M}$ with the fibre $\pi^{-1}(J)$ at a point $J \in \mathcal{M}$ being isomorphic to the quantum Hilbert space H_J constructed with the help of the complex structure J.

The tangent space of the supermanifold Superdiff $S^1/\text{OSp}(1|2)$ is spanned by vector fields K_n, H_r with $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$, which at the origin satisfy the commutation superalgebra (2.18). If one defines on the Fock vacuum bundle $\mathcal{B}$ the connection

$$\nabla_{K_n} = \mathcal{L}_{K_n} + r(\lambda_n), \qquad \nabla_{H_r} = \mathcal{L}_{H_r} + r(\varphi_r), \tag{2.28}$$

then it is easy to show that this connection leaves the hermitian inner product (2.26) on $\mathcal{B}$ invariant if and only if [3]

$$r(\lambda_{-n}) = -[r(\lambda_n)]^+, \qquad r(\varphi_{-r}) = [r(\varphi_r)]^+,$$

where the superscript + denotes hermitian conjugation. These expressions define the operators $r(\lambda_n)$ and $r(\varphi_r)$ for $n<-1$ and $r<-\frac{1}{2}$. It is worth pointing out that

$$\mathcal{O}_{\lambda_{-1}} = -[\mathcal{O}_{\lambda_1}]^+ , \qquad \mathcal{O}_{\varphi_{-1/2}} = [\mathcal{O}_{\varphi_{1/2}}]^+ .$$

The curvature of the connection (2.28) is given by [3,4]

$$\begin{aligned} F(K_{-m}, K_n) &= [\nabla_{K_{-m}}, \nabla_{K_n}] - \nabla_{[K_{-m}, K_n]} \\ &= [r(\lambda_{-m}), r(\lambda_n)] + (m+n) r(\lambda_{-m+n}) \\ &= \tfrac{1}{8} d(m^3 - m)\delta_{m,n} , \\ F(H_{-r}, H_s) &= [r(H_{-r}), r(H_s)] - 2r(K_{-r+s}) \\ &= \tfrac{1}{8} d(4r^2 - 1)\delta_{r,s} , \end{aligned}$$

where d is the dimension of space–time.

2.4. INVARIANT VACUA OF THE NEVEU–SCHWARZ SUPERSTRING

According to refs. [2–4], Superdiff S^1 invariant vacua for the Neveu–Schwarz superstring may be represented by antiholomorphic and covariantly constant sections of the tensor product bundle $\mathcal{B}\otimes\bar{\Gamma}$ over the supermanifold, $\mathcal{M}=\text{Superdiff } S^1/\text{OSp}(1|2)$, of all complex structures over $\Omega\mathbb{M}^{d|d}_{\text{NS}}$, where $\mathcal{B}$ is the Fock vacuum bundle and $\bar{\Gamma}$ the anticanonical line bundle over $\mathcal{M}$.

Thus the condition necessary to define a Superdiff S^1 invariant superstring vacuum is the vanishing of the total curvature,

$$\begin{aligned} F(K_{-m}, K_n) + \text{Ric}(K_-, K_n) &= \tfrac{1}{8}(d-10)(m^3 - m)\delta_{m,n} , \\ F(H_{-r}, H_s) + \text{Ric}(H_{-r}, H_s) &= \tfrac{1}{8}(d-10)(4r^2 - 1)\delta_{r,s} , \end{aligned}$$

of the bundle $\mathcal{B}\otimes\bar{\Gamma}$. This is possible if and only if $d=10$. Thus it is the flatness of the tensor product bundle $\mathcal{B}\otimes\bar{\Gamma}$ that yields the critical dimension of 10 for the Neveu–Schwarz superstring in the geometric quantization approach.

3. Geometric quantization of the Ramond superstring

Let us denote by $\mathcal{L}\mathbb{M}^{d|d}_{\text{R}}$ the space of all maps

$$x\oplus\psi : [0, 2\pi] \to \mathbb{M}^{d|d} \equiv \mathbb{M}^{d|0} \oplus \mathbb{M}^{0|d}$$

with periodic boundary conditions in the bosonic coordinates and periodic boundary conditions in the fermionic coordinates. An element of $\mathcal{L}\mathbb{M}^{d|d}_{\text{R}}$ is a pair (x^μ, ψ^μ) consisting of a bosonic function $x^\mu(\sigma)$ and a fermionic function $\psi^\mu(\sigma)$

which satisfy the boundary conditions $x^\mu(0)=x^\mu(2\pi)$, $\psi^\mu(0)=\psi^\mu(2\pi)$, $\mu=1, ..., d$. We also define the space, $\Omega\mathbb{M}_{\mathbb{R}}^{d|d}$, of based superloops as the quotient space $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}/\mathbb{M}^{d|0}$. The space $\Omega\mathbb{M}_{\mathbb{R}}^{d|d}$ may be identified with the subspace of $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}$ consisting of all superloops (x, ψ) with bosonic loop x beginning and ending at the origin of $\mathbb{M}^{d|0}$. It has been shown in ref. [3] that the space $\Omega\mathbb{M}_{\mathbb{R}}^{d|d}$ may be interpreted either as the configuration space for closed Ramond superstrings or as the phase space for open Ramond superstrings.

Let us consider the vector fields

$$L_m = -\mathrm{i}\int \mathrm{d}\sigma\, \mathrm{e}^{\mathrm{i}m\sigma}\left[\frac{\mathrm{d}x^\mu(\sigma)}{\mathrm{d}\sigma}\frac{\delta}{\delta x^\mu(\sigma)} + \left(\frac{\mathrm{d}\psi^\mu(\sigma)}{\mathrm{d}\sigma} + \mathrm{i}\frac{m}{2}\psi^\mu(\sigma)\right)\frac{\delta}{\delta\psi^\mu(\sigma)}\right],$$

$$G_r = \int \mathrm{d}\sigma\, \mathrm{e}^{\mathrm{i}r\sigma}\left(\psi^\mu(\sigma)\frac{\delta}{\delta x^\mu(\sigma)} + \mathrm{i}\frac{\mathrm{d}x^\mu(\sigma)}{\mathrm{d}\sigma}\frac{\delta}{\delta\psi^\mu(\sigma)}\right), \tag{3.1}$$

on $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}$, where both m and r range over the set of integers $\mathbb{Z}$. They satisfy the commutation superalgebra

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [L_m, G_r] = (\tfrac{1}{2}m - r)G_{m+r},$$
$$[G_r, G_s] = 2L_{r+s}, \quad m, n, r, s \in \mathbb{Z},$$

and provide thus a representation of the superalgebra $\underline{\underline{\text{Superdiff}}}\, S^1$ on $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}$ [cf. (2.1), (2.2)].

If we expand the loops $x^\mu(\sigma)$ and $\psi^\mu(\sigma)$ in their normal modes,

$$x^\mu(\sigma) = \sum_{m\in\mathbb{Z}} x_m^\mu \mathrm{e}^{\mathrm{i}m\sigma},$$
$$\psi^\mu(\sigma) = \sum_{r\in\mathbb{Z}} \psi_r^\mu \mathrm{e}^{\mathrm{i}r\sigma},$$

then the set of modes $\{x_n^\mu, \psi_r^\mu, n\in\mathbb{Z}, r\in\mathbb{Z}\}$ provides a natural coordinate system on $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}$, while the set of modes $\{x_n^\mu, \psi_r^\mu, n\in\mathbb{Z}\setminus 0, r\in\mathbb{Z}\}$ provides a coordinate system on $\Omega\mathbb{M}_{\mathbb{R}}^{d|d}$.

There is a natural symplectic metric on $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}$ [3],

$$\omega(U, V) = \frac{1}{2\pi}\int \eta_{\mu\nu}\left(U^\mu(\sigma)\frac{\mathrm{d}}{\mathrm{d}\sigma}V^\nu(\sigma) - \mathrm{i}\tilde{U}^\mu(\sigma)\tilde{V}^\nu(\sigma)\right)\mathrm{d}\sigma$$
$$= \mathrm{i}\sum_{m\in\mathbb{Z}} mU_{-m}^\mu V_m^\nu \eta_{\mu\nu} - \mathrm{i}\sum_{r\in\mathbb{Z}} \tilde{U}_r^\mu \tilde{V}_{-r}^\nu \eta_{\mu\nu}. \tag{3.2}$$

Though this metric is obviously degenerate on $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}$, it can be regarded as a lift to $\mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}$ of a non-degenerate symplectic metric on $\Omega\mathbb{M}_{\mathbb{R}}^{d|d}$, which we denote by the same symbol ω. Therefore the quotient superspace $\Omega\mathbb{M}_{\mathbb{R}}^{d|d} = \mathscr{L}\mathbb{M}_{\mathbb{R}}^{d|d}/\mathbb{M}^{d|0}$

comes equipped with a natural symplectic structure. The vector fields (3.1) on $\mathscr{L}\mathbb{M}_{\mathrm{R}}^{d|d}$ are mapped under the projection $\mathscr{L}\mathbb{M}_{\mathrm{R}}^{d|d} \to \Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ into vector fields $L_m \operatorname{mod}(\partial/\partial x_0^\mu)$, $G_r \operatorname{mod}(\partial/\partial x_0^\mu)$ on $\Omega\mathbb{M}_{\mathrm{NS}}^{d|d}$, which satisfy the same commutation superalgebra (3.2) and are denoted by the same symbols.

Up to now our discussion of the Ramond superstring theory simply mimics the previous discussion of the Neveu–Schwarz superstring. If, however, we attempt naively to define a complex structure on $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ by the same formulae (2.16) as in the case of the Neveu–Schwarz superstring theory, we immediately face the problem of defining the action of J on the fermionic zero mode ψ_0^μ. In ref. [3] it was proposed to solve this problem by using the quotient space $\mathscr{L}\mathbb{M}_{\mathrm{R}}^{d|d}/\mathbb{M}^{d|d}$ instead of $\Omega\mathbb{M}_{\mathrm{R}}^{d|d} = \mathscr{L}\mathbb{M}_{\mathrm{R}}^{d|d}/\mathbb{M}^{d|0}$. This proposal is, however, incorrect, since the natural symplectic metric (3.2) on $\mathscr{L}\mathbb{M}_{\mathrm{R}}^{d|d}$ is *not* a lift of a non-degenerate symplectic metric on $\mathscr{L}\mathbb{M}_{\mathrm{R}}^{d|d}/\mathbb{M}^{d|d}$. The calculations in ref. [3] were actually performed for objects defined on $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ rather than on $\mathscr{L}\mathbb{M}_{\mathrm{R}}^{d|d}/\mathbb{M}^{d|d}$.

Thus in order to define a complex structure on $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ we have to define an almost complex structure J_0 on $\mathbb{M}^{0|d}$. We suppose from now on that $d=2k$ for some $k\in\mathbb{Z}$ and $\mathbb{M}^{d|0}$ is a Euclidean space. The latter supposition is taken only for simplicity with the motivation that in the Euclidean case it turns out to be sufficient to use solely Kähler polarizations in the geometric quantization scheme. The case of space–time with flat Lorentzian metric can be equally well studied by methods of geometric quantization, though it requires the use of a slightly more complicated polarization of mixed type.

We define J_0 by writing

$$J_0(\partial/\partial\psi_0^{2a}) = \partial/\partial\psi_0^{2a-1}\,, \qquad J_0(\partial/\partial\psi_0^{2a-1}) = -\,\partial/\partial\psi_0^{2a}\,, \tag{3.3}$$

where $a=1, \ldots, \frac{1}{2}d$. Then we define the following complex structure on $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$:

$$\begin{aligned} J(W) = &-\mathrm{i}\sum_{n\neq 0} \operatorname{sgn}(n)\; W_n^\mu\, \partial/\partial x_n^\mu - \mathrm{i}\sum_{r\neq 0} \operatorname{sgn}(r)\; \tilde{W}_r^\mu\, \partial/\partial\psi_r^\mu \\ &- \tilde{W}_0^\mu J_0(\partial/\partial\psi_0^\mu)\,, \end{aligned} \tag{3.4}$$

where

$$W = \sum_{n\neq 0} W_n^\mu\, \partial/\partial x_n^\mu + \sum_{r\in\mathbb{Z}} \tilde{W}_r^\mu\, \partial/\partial\psi_r^\mu$$

is an arbitrary vector on $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$.

The complex structure (3.4) on $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ determines a Kähler polarization (see appendix). The subspace $T^{1,0}\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ of the complexified tangent bundle over $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ is spanned by all the vectors $\{\partial/\partial x_n^\mu, \partial/\partial\psi_r^\mu, \partial/\partial\zeta^a = \partial/\partial\psi_0^{2a-1} - \mathrm{i}\,\partial/\partial\psi_0^{2a}$, with $n>0$, $r>0$ and $a=1, \ldots, \frac{1}{2}d\}$. Thus the quantum Hilbert space H consists of functionals $\Psi\colon \Omega\mathbb{M}_{\mathrm{R}}^{d|d} \to \mathbb{C}$ which depend only on the coordinates x_{-n}^μ, ψ_{-r}^μ and $\zeta^a \equiv \psi_0^{2a-1} - \mathrm{i}\psi_0^{2a}$ with $n, r>0$ and $a=1, \ldots, \frac{1}{2}d$.

At first sight it might appear that the constructed quantum space H is inconsistent with SO(d) invariance. Fortunately this is not so. From the Clifford algebra description of spinors (see e.g. refs. [5,9]) it follows that the space H furnishes a *spinor* representation of the space–time group SO(d). Thus within the geometric quantization approach to the Ramond superstring it is the complex structure (3.4) that is responsible for the space–time fermionic nature of superstring ground states. This result is in full accord with the usual operator approach to quantization of this theory [5].

Using the same methods as in subsection 2.1 one may check that the complex structure (3.4) is invariant under the actions of the vector fields L_0 and G_0. This result implies that the supermanifold $\mathscr{M}$ of all complex structures on $\Omega\mathbb{M}_{\mathrm{R}}^{d|d}$ which are connected to each other by Superdiff S^1 is isomorphic to the homogeneous supermanifold Superdiff S^1/Super S^1 just as in refs. [3,4].

The rest of the geometric quantization procedure is the same as in refs. [3,4]. The curvature of the Fock vacuum bundle over Superdiff S^1/Super S^1 is given by [3,4]

$$F(K_{-m}, K_n) = \tfrac{1}{8} d m^3 \delta_{m,n}\,, \qquad F(H_{-r}, H_s) = \tfrac{1}{2} d r^2 \delta_{r,s}\,,$$

while the Ricci curvature of Superdiff S^1/Super S^1 has the form [3,4]

$$\mathrm{Ric}(K_{-m}, K_n) = -\tfrac{10}{8} m^3 \delta_{m,n}\,, \qquad \mathrm{Ric}(G_{-r}, G_s) = -5 r^2 \delta_{r,s}\,.$$

Superdiff S^1 invariant vacua for the Ramond superstring are represented by antiholomorphic and covariantly constant sections of the tensor product bundle $\mathscr{B}\otimes\bar{\Gamma}$ over Superdiff S^1/Super S^1. The condition necessary to define such a section is the vanishing of the total curvature,

$$F(K_{-m}, K_n) + \mathrm{Ric}(K_{-m}, K_n) = \tfrac{1}{8}(d-10)(m^3 - m)\delta_{m,n}\,,$$

$$F(H_{-r}, H_s) + \mathrm{Ric}(H_{-r}, H_s) = \tfrac{1}{8}(d-10)(4r^2 - 1)\delta_{r,s}\,,$$

of the bundle $\mathscr{B}\otimes\bar{\Gamma}$. This is possible if and only if $d=10$. Thus the only novelty the complex structure (3.4) brings into the holomorphic description of Ramond superstrings is that the corresponding ground states furnish a spinor representation of SO(d).

A final remark is in order. Nag and Verjovsky [10] proved the Kähler manifold Diff S^1/SL(2, R) is a dense complex supermanifold of the Universal Teichmüller space which contains all the Teichmüller spaces $\mathscr{T}_g$, $g \geqslant 1$, corresponding to compact Riemann surfaces of genus g. If non-perturbative bosonic string amplitudes can indeed be written as integrals over Diff S^1/SL(2, R) with the measure determined by the canonical Kähler structure, then the result of Nag and Verjovsky gives an instrument for comparing non-perturbative string theory with the standard sum-over-moduli approach. It is natural to suggest that the supermanifolds Superdiff S^1/OSp(1|2) and Superdiff S^1/Super S^1 are analogously related to the "Universal Teichmüller superspace" associated with supermoduli of SUSY curves.

Appendix A. Review of geometric quantization

The goal of the programme of geometric quantization [11,12] is to find a "natural" and rigorous way to associate a quantum Hilbert space, H, to a phase space of some classical systems. The latter is described by a symplectic manifold, i.e. a pair (M, ω), consisting of a $2n$-dimensional manifold M, together with a non-degenerate closed two-form, ω, on M. The ring of functions, $\mathcal{A}_M$, on M comes equipped with a Lie algebra structure represented by the Poisson bracket

$$\{f, g\} = -\omega(X_f, X_g) , \tag{A.1}$$

where f and g are arbitrary functions on M, and $X_f \equiv -\omega^{-1}(\mathrm{d}f)$, $X_g \equiv -\omega^{-1}(\mathrm{d}g)$ are the corresponding Hamiltonian vector fields, which, as one may check, satisfy the relation

$$[X_f, X_g] = X_{\{f,g\}} . \tag{A.2}$$

A prequantization of the classical system (M, ω) under consideration is a representation of the Lie algebra, $\mathcal{A}_M$, of functions on M ("observables") in some *prequantum* Hilbert space $\tilde{H}$, i.e. a linear mapping $\mathcal{O}: \mathcal{A}_M \to \mathcal{A}_{\tilde{H}}, f \to \mathcal{O}_f$, from $\mathcal{A}_M$ to the Lie algebra, $\mathcal{A}_{\tilde{H}}$, of operators acting on $\tilde{H}$. It is supposed that this mapping satisfies the condition

$$[\mathcal{O}_f, \mathcal{O}_g] = -\mathrm{i}\,\mathcal{O}_{\{f,g\}} \tag{A.3}$$

for any $f, g \in \mathcal{A}_M$.

The first element of the geometric quantization scheme is a hermitian line bundle $p: L \to M$ equipped with a connection ∇ leaving the hermitian scalar product invariant. If round brackets (,) denote the hermitian scalar product, if Ψ and Φ are smooth sections of L, and if X is a vector field on M, the invariance condition reads

$$X(\Psi, \Phi) = (\nabla_X \Psi, \Phi) + (\Psi, \nabla_X \Phi) . \tag{A.4}$$

If the symplectic form ω has integral periods, then there exists a hermitian line bundle $L \to M$ equipped with a connection ∇ such that its curvature tensor is given by

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = -\mathrm{i}\omega(X, Y) . \tag{A.5}$$

The space of smooth sections of this bundle is a "natural" candidate for the vacant position of a prequantum Hilbert space $\tilde{H}$ with the inner product defined by

$$\langle \Psi, \Phi \rangle = \int (\Psi, \Phi) \det \omega , \tag{A.6}$$

and the following explicit operator realization of a function $f \in \mathcal{A}_M$:

$$f \to \mathcal{O}_f = -\mathrm{i}\, \nabla_{X_f} + f\,, \tag{A.7}$$

where X_f is the Hamiltonian vector field associated with f. It is an easy exercise to check that the operators (A.7) do satisfy the requirement (A.3).

The constructed prequantum Hilbert space $\tilde{H}$ violates, however, the uncertainty principle (the "wave functions" Ψ and Φ depend on both the "momentum" and "position" variables). Therefore, in order to describe quantum mechanics one has to find a "natural" way to eliminate n degrees of freedom. We shall assume, from now on, that M is a Kähler manifold, i.e., it admits both a symplectic structure ω and an integrable complex structure J which satisfy the condition that the metric defined by

$$g(X, Y) = \omega(X, JY) \tag{A.8}$$

is hermitian. The complex structure J splits the complexified tangent bundle $\mathbb{C}\otimes_{\mathbb{R}} TM$ into a direct sum,

$$\mathbb{C}\otimes_{\mathbb{R}} TM = T^{1,0}M + T^{0,1}M\,,$$

of complex subspaces with the property

$$J|_{T^{1,0}M} = -\mathrm{i}\ \mathrm{id}|_{T^{1,0}M}\,, \qquad J|_{T^{0,1}M} = \mathrm{i}\ \mathrm{id}|_{T^{0,1}M}\,.$$

The integrability of the complex structures is equivalent to the closedness of the n-dimensional distribution $T^{1,0}M$ under commutation:

$$[V, W] \bmod T^{1,0}M = 0\,,$$

for any $V, W \in T^{1,0}M$.

The second element of the geometric quantization scheme is the quantum Hilbert space, H, the space of polarized sections. By definition, this space is a subspace of $\tilde{H}$ consisting of those sections, Ψ, of the line bundle L which satisfy the polarization condition

$$\mathcal{L}_V \Psi = 0 \tag{A.9}$$

for any vector field $V \in T^{1,0}M$.

If a classical observable f is such that its Hamiltonian vector field satisfies the condition

$$\mathcal{L}_V X_f \,(\bmod\ T^{1,0}M) = 0$$

for any $V \in T^{1,0}M$, then the prequantum operator $\mathcal{O}_f$ given by (A.7) is also appropriate at the quantum level, since it preserves the polarization condition and defines a map $\mathcal{O}_f : H \to H$.

Since M is a $2n$-dimensional Kähler manifold, it can be covered by complex coordinate charts $\{z^m, m = 1, \ldots, n\}$ in which the symplectic form takes the form

$$\omega_{mn} = -\mathrm{i}\, \partial_m \partial_{\bar{n}} K\,,$$

with K being the Kähler potential and $\partial_m = \partial/\partial z^m$, $\partial_{\bar{n}} = \partial/\partial z^{\bar{n}}$. There always exists a gauge in which the corresponding covariant derivatives on L are given by

$$\nabla_m = \partial_m , \qquad \nabla_n = \partial_{\bar{n}} - \partial_{\bar{n}} K .$$

It is easy to check that these operators leave invariant the hermitian product

$$(\Psi, \Phi) = \bar{\Psi}\Phi e^{-K}$$

on L. The polarization condition takes now the form

$$\partial_m \Psi = 0 ,$$

which implies that the quantum Hilbert space H consists of antiholomorphic sections of the line bundle L which are normalizable with respect to the inner product

$$\langle \Psi, \Phi \rangle = \int \bar{\Psi}\Phi \, e^{-K} \det \omega . \qquad \text{(A.10)}$$

References

[1] M.J. Bowick and S.G. Rajeev, The holomorphic geometry of closed bosonic string and Diff S^1/S^1, Nucl. Phys. B 293 (1987) 348–384.
[2] S.A. Merkulov, Non-perturbative bosonic string and Diff S^1/SL(2, R), JETP Lett. 55 (1992) 153–155.
[3] D. Harari, D.K. Hong, P. Ramond and V. Rodgers, The superstring, Diff S^1/S^1, and holomorphic geometry, Nucl. Phys. B 294 (1987) 556–572.
[4] Z.Y. Zhao, K. Wu and T. Saito, Phys. Lett. B 199 (1987) 37.
[5] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory* (Cambridge Univ. Press, 1987).
[6] Yu.I. Manin, *Complex Geometry and Gauge Field Theory* (Nauka, Moscow, 1984).
[7] S. Kobayashi and K. Nomizi, *Foundations of Differential Geometry*, Vol. 2 (Interscience, New York, 1969).
[8] D. Freed, in: *Infinite Dimensional Group with Applications*, ed. V. Kac (Springer, Berlin, 1985).
[9] Th. Voronov, in: *Modern Problems of Mathematics*, Vol. 38 (Viniti, Moscow, 1991), in Russian.
[10] S. Nag and A. Verjovsky, Diff S^1/SL(2, R) and Teichmüller spaces, Commun. Math. Phys. 130 (1990) 123.
[11] N.J.M. Woodhouse, *Geometric Quantization* (Clarendon, 1980).
[12] J. Sniatycki, *Geometric Quantization and Quantum Mechanics* (Springer, Berlin, 1980).